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# DECOMPOSITION OF RANGES OF VECTOR MEASURES

BY

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ABSTRACT

The following conditions on a zonoid $Z$, i.e., a range of a non-atomic vector measure, are equivalent: (i) the extreme set containing 0 in its relative interior is a parallelepiped; (ii) the zonoid $Z$ determines the $m$-range of any non-atomic vector measure with range $Z$, where the $m$-range of a vector measure $\mu$ is the set of $m$-tuples $(\mu(S_1), \cdots, \mu(S_m))$, where $S_1, \cdots, S_m$ are disjoint measurable sets; and (iii) there is a vector measure space $(X, \Sigma, \mu)$ such that any finite factorization of $Z$, $Z = \Sigma Z_i$, in the class of zonoids could be achieved by decomposing $(X, \Sigma)$. In the case of ranges of non-atomic probability measures (i) is automatically satisfied, so (ii) and (iii) hold.

## 1. Introduction

Let $\Sigma$ be a $\sigma$-field of subsets of a set $X$ and $\mu : \Sigma \to R^n$ be countably additive. Such a $\mu$ is called a *vector measure*; $\mu$ is *non-atomic* if $\mu(E) \neq 0$ implies the existence of an $F \subseteq E$ with $0 \neq \mu(F)$ and $\mu(F) \neq \mu(E)$. A *zonoid* is the range $\mathcal{R}(\mu) = \{\mu(E) \mid E \in \Sigma\}$ of a non-atomic vector measure $\mu$. Write $\mathcal{F}$ for the set of zonoids. Lyapunov's theorem asserts that $\mathcal{F} \subset \mathcal{K}$, where $\mathcal{K}$ is the class of all convex compact subsets of $R^n$ which contain 0.

The *decomposition m-range* $\mathcal{R}_m(\mu)$ of the vector measure $\mu$ is defined to be the family of $m$ $n$-dimensional vectors

$$\mathcal{R}_m(\mu) = \left\{ (\mu(S_1), \cdots, \mu(S_m)) : \bigcup_{i=1}^{m} S_i = X, S_i \in \Sigma, S_i \cap S_j = \emptyset \text{ for } i \neq j \right\}.$$

Let $Z \in \mathcal{F}$ and let $\mu$ be a non-atomic vector measure with $\mathcal{R}(\mu) = Z$. We say that $Z = Z_1 + Z_2$ is a *zonoid decomposition* of $Z = \mathcal{R}(\mu)$ if $Z_1, Z_2 \in \mathcal{F}$. It is a *zonoid decomposition with respect to* $\mu$ if there is $S$ in $\Sigma$ such that $Z_1 = \mathcal{R}(\mu, S) \equiv \{\mu(E) \mid E \in \Sigma, E \subset S\}$. Observe that then also $Z_2 = \mathcal{R}(\mu, S^c)$ where $S^c = X \backslash S$.

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A compact convex subset $E$ of a compact convex set $K$ is an extreme set of $K$ if $\alpha x + (1-\alpha)y \in E$ and $0 < \alpha < 1$, $x, y \in K$ imply that $x \in E$. For every $x \in K \in \mathcal{K}$ there is a unique extreme set $E$ of $K$ containing $x$ in its relative interior. The set of extreme points of a compact convex set $K$ is denoted by ext $K$.

THEOREM I. *The following conditions on a zonoid $Z$ are equivalent.*

(A) *For every two non-atomic vector measures $\mu$ and $\sigma$, with $\mathcal{R}(\mu) = \mathcal{R}(\sigma) = Z$, $\mathcal{R}_m(\mu) = \mathcal{R}_m(\sigma)$ for every $m \geqq 1$.*

(B) *There is a vector measure space $(X, \Sigma, \mu)$ with $\mathcal{R}(\mu) = Z$, such that any zonoid decomposition $Z = Z_1 + Z_2$ is a decomposition with respect to $\mu$.*

(C) *The extreme set of $Z$ containing $0$ in its relative interior is a parallelepiped.*

In Section 2 we define a *definite zonoid.*

The proof of Theorem I is accomplished (in Section 3) by proving that each of the three conditions, (A), (B) and (C), is equivalent to a fourth one, (D) — $Z$ is a definite zonoid.

Along the proof of Theorem I we obtain some results which might have independent interest. In Section 4 we study decompositions of sums of countable many one-dimensional zonoids.

## 2. The standard measure

In this section we sketch briefly the relation between the range of a vector measure and the distribution of the Radon–Nikodym derivatives of its components. A more detailed account is given in Bolker [1], where additional results and references are given.

Let $(X, \Sigma, \mu)$ be a vector measure space; $\mu : \Sigma \to R^n$. Let $\|\mu\|_2$, or $|\mu|$ for short, be the total variation of $\mu$ with respect to the Euclidean norm, i.e., $|\mu|$ is the scalar measure on $(X, \Sigma)$ given by

$$|\mu|(S) = \sup \sum_{i=1}^{n} \|\mu(T_i)\|_2,$$

where the sup is taken over all measurable partitions $(T_i)_{i=1}^n$ of $X$, i.e., $T_i \in \Sigma$, $T_i \cap T_j = \emptyset$, for $1 \leqq i < j \leqq n$ and $\bigcup_{i=1}^n T_i = X$.

Let $f$ be the Radon–Nikodym derivative of $\mu$ with respect to $|\mu|$. Then $f : X \to S^{n-1}$, $|\mu|$-almost everywhere. Denote by $\eta_\mu$ the measure $|\mu| \circ f^{-1}$ on $(S^{n-1}, \mathcal{B})$ where $\mathcal{B}$ denotes the Borel subsets of $S^{n-1}$. A positive scalar measure $\eta$ on $S^{n-1}$ will be called a *standard measure.* Every standard measure $\eta$ induces a vector measure $\tilde{\eta}$ on $S^{n-1}$ given by $\tilde{\eta}(A) = \int_A u d\eta(u)$ for every $A \in \mathcal{B}$. The vector measure $\tilde{\eta}$ is non-atomic if and only if $\eta$ is non-atomic. The convex hull of the range of a vector measure is the range of a non-atomic vector measure, and thus the convex hull of $\mathcal{R}(\tilde{\eta})$, which will be denoted by $Z(\eta)$, is a zonoid. If $\eta$ is a measure on $R^n$ with $\int \|x\| d\eta(x) < \infty$, $Z(\eta)$ is similarly defined. If $\eta_1, \eta_2$ are two standard measures then $Z(\eta_1 + \eta_2) = Z(\eta_1) + Z(\eta_2)$. The support function $H(K, \cdot)$ of a convex set $K$ is defined for $\xi \in R^n$ by $H(K, \xi) = \sup\{\langle x, \xi \rangle \mid x \in K\}$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $R^n$. If $\eta$ is the standard measure associated to the non-atomic vector measure, then for every $A \in \mathcal{B}$ $\int_A u \cdot d\eta(u) = \mu(f^{-1}(A))$, and thus $\mathcal{R}(\mu) \supset \mathcal{R}(\tilde{\eta})$. On the other hand, $H(\mathcal{R}(\mu), \xi) = \langle \mu(A(\xi)), \xi \rangle$, where $A(\xi) = \{x \in X \mid \langle f(x), \xi \rangle \geqq 0\} = f^{-1}(H_\xi)$, where $H_\xi = \{u \in S^{n-1} \mid \langle u, \xi \rangle \geqq 0\}$, and thus $Z(\eta) \supset \mathcal{R}(\mu)$. As $\mathcal{R}(\mu)$ is convex, and $\mathcal{R}(\tilde{\eta})$ closed, $Z(\eta) = \mathcal{R}(\mu)$. Thus every zonoid is of the form $Z(\eta)$ for some standard measure $\eta$. If $\eta$ is a standard measure we define the standard measure $\eta^s$ on $S^{n-1}$ by $\eta^s(E) = (\eta(E) + \eta(-E))/2$.

LEMMA 2.1. ([4, theorem 2], [2, theorem 22], [1, corollary 2.9]). *$Z(\eta_1)$ is a translate of $Z(\eta_2)$ if and only if $\eta_1^s = \eta_2^s$.*

A zonoid $Z$ is called a *definite zonoid* if it uniquely determines its standard measure, i.e., for any two standard measures $\eta_1$ and $\eta_2$ with $Z(\eta_1) = Z = Z(\eta_2)$, $\eta_1 = \eta_2$.

## 3. The proof of Theorem 1

We will prove that each of the conditions (A), (B), (C) is equivalent to (D) — $Z$ is definite.

LEMMA 3.1. *Let $Z$ be a definite zonoid, 0 the center of symmetry of $Z$. Then $Z$ is a parallelepiped.*

PROOF. Let $Z = Z(\mu)$. If $\mu^*$ denotes the standard measure given by $\mu^*(E) = \mu(-E)$ then $Z(\mu^*) = -Z(\mu)$. As 0 is the center of symmetry of $Z(\mu)$, $Z(\mu^*) = -Z(\mu) = Z(\mu)$ and thus $Z(\mu^s) = Z(\mu)$, but $Z$ is definite and therefore $\mu = \mu^s$.

Let $T: L_\infty(\mu) \to R^n$ be the linear transformation defined by $Tf = \int_{S^{n-1}} f(u) u d\mu(u)$.

Let $W$ be the subspace of $L_\infty(\mu)$ of all anti-symmetric functions, i.e., $f(x) = -f(-x)$ $\mu$-almost everywhere.

Assume first that $\dim W > n$. Then, there is $f \in W$, $0 < \|f\|_\infty < 1$, such that $Tf = 0$, i.e., $\int f(x) x d\mu(x) = 0$. Let $\eta$ be the measure on $S^{n-1}$ which is absolutely continuous with respect to $\mu$, with $d\eta/d\mu = 1-f$. As $\|f\|_\infty < 1$, $\eta$ is a standard measure, and as $\|f\| \neq 0$, $\eta \neq \mu$. As $f$ is anti-symmetric it follows that for every $A \in \mathcal{B}$, $\eta(A \cup -A) = \mu(A \cup -A)$ and therefore by Lemma 2.1, $Z(\eta)$ is a translate of $Z(\mu)$. In order to show that $Z(\eta) = Z(\mu)$ we have to show that both have the same center of symmetry, i.e., to show that $\tilde{\eta}(S^{n-1}) = \tilde{\mu}(S^{n-1})$. But

$$\tilde{\eta}(S^{n-1}) = \int (1-f)(x)x d\mu(x) = \tilde{\mu}(S^{n-1}) - \int f(x)x d\mu(x) \tilde{\mu}(S^{n-1}) = \tilde{\mu}(S^{n-1})$$

and thus $Z(\eta) = Z(\mu)$. Therefore if $Z$ is a definite zonoid, $\dim W \leqq n$. However, $\dim W \leqq n$ iff there is a finite set $A$ with $\# A \leqq n$, such that $A \cup -A$ is a support of $\mu$. Without loss of generality we could assume that $\dim Z = n$ which implies that the set $A$ consists of linearly independent vectors, i.e., $Z(\mu)$ is a sum of linearly independent line segments — a parallelepiped.

Lemma 3.2. *Every zonoid $Z$ has a decomposition of the form $Z = Z_1 + Z_2$ where 0 is the center of symmetry of $Z_1$ and an extreme point of $Z_2$.*

Proof. Let $Z = Z(\mu)$ where $\mu$ is a standard measure, $v = \int x d\mu(x)$. Denote by $W'$ the set $W' = \{f \in L_\infty(\mu) \mid 0 \leqq f \leqq 1, \int f(x)x d\mu(x) = v\}$, i.e., in the notations of the previous lemma, $W' = T^{-1}(v) \cap \{f \in L_\infty(\mu) \mid 0 \leqq f \leqq 1\}$. Thus $W'$ is convex and compact, and if $f_n \in W'$, $f_{n+1} \leqq f_n$ and $\lim f_n = f$ then by the Lebesgue dominated convergence theorem $f \in W'$. Using Zorn's lemma we deduce the existence of $\bar{f} \in W'$ such that there is no $g \in W'$ with $g \leqq \bar{f}$, $g \neq \bar{f}$. Let $\mu_1$ be the standard measure whose Radon–Nikodym derivative with respect to $\mu$ is $\bar{f}$, and $\mu_2 = \mu - \mu_1$. Then $Z(\mu) = Z(\mu_1) + Z(\mu_2)$. First we claim that $0 \in \operatorname{ext} Z(\mu_1)$. Otherwise there is $0 \leqq g_1, g_2 \leqq \bar{f}$ such that $0 \neq \int g_1(x)x d\mu(x) = -\int g_2(x)x d\mu(x)$. Let $f' = \bar{f} - (g_1 + g_2)/2$. Then $f' < f$, and $f' \in W'$. Thus $0 \in \operatorname{ext} Z(\mu_1)$. As $\bar{f} \in W'$, $\tilde{\mu}_1(S^{n-1}) = v$ and thus $\tilde{\mu}_2(S^{n-1}) = v - v = 0$, which proves that 0 is the center of symmetry of $Z_2$.

Lemma 3.3. *Let 0 be an extreme point of a zonoid $Z$. Then $Z$ is definite.*

Proof. By induction on $\dim Z$. Assume $\eta_1$ and $\eta_2$ are two standard measures with $Z = Z(\eta_1) = Z(\eta_2)$. We have to prove that $\eta_1 = \eta_2$. If $\dim Z = 1$, this is obvious. Let $\dim Z > 1$. As 0 is an extreme point of $Z$, there is $\xi \in R^n$ for which $\langle x, \xi \rangle \leqq 0$ for every $x \in Z$, and $\dim F(Z, \xi) < \dim Z$ where $F(Z, \xi)$ is the corresponding face, i.e., $F(Z, \xi) = \{x \in Z \mid \langle x, \xi \rangle = H(Z, \xi) = 0\}$. Then $H_\xi$ is a support for $\eta_i$. If we denote by $\bar{\eta}_i$, $i = 1, 2$, the standard measure $\eta_i$ restricted to $\xi^\perp = \{x \in R^n \mid \langle x, \xi \rangle = 0\}$, then $F(Z, \xi) = Z(\bar{\eta}_i)$. By the induction hypothesis $\bar{\eta}_1 = \bar{\eta}_2$. Let $\sigma_i = \eta_i - \bar{\eta}_i$, $i = 1, 2$. Then as $Z(\eta_i) = Z(\bar{\eta}_i) + Z(\sigma_i)$ we conclude that $Z(\sigma_1)=Z(\sigma_2)$. Thus, by Lemma 2.1, $\sigma_1^s=\sigma_2^s$. As $H_\xi^0\equiv H_\xi^0\backslash\xi^\perp$ is a support of both $\sigma_1$ and $\sigma_2$ and as $H_\xi^0\cap -H_\xi^0=\varnothing$ we conclude that $\sigma_1=\sigma_2$ and thus $\eta_1=\eta_2$.

Two sets $K_1,K_2\in K$ are independent if $\dim(K_1+K_2)=\dim K_1+\dim K_2$, where $\dim K$ (for $K\in\mathcal{K}$) is the dimension of the minimal subspace containing $K$.

LEMMA 3.4. *If $Z_1$ and $Z_2$ are two independent definite zonoids then $Z_1+Z_2$ is definite.*

PROOF. Let $\eta$, $\eta_1$, $\eta_2$ be standard measures with $Z(\eta)=Z(\eta_1)+Z(\eta_2)$ and $Z(\eta_i)=Z_i$, $i=1,2$. Then $Z(\eta^s)=Z(\eta_1^s)+Z(\eta_2^s)=Z(\eta_1^s+\eta_2^s)$ and thus by Lemma 2.1, $\eta^s=\eta_1^s+\eta_2^s$. Let $V_i$, $i=1,2$, be the minimal subspace of $R^n$ which contains $Z_i$. As $Z_1$ and $Z_2$ are independent, $V_1\cap V_2=\{0\}$ and thus $V_1\cap V_2\cap S^{n-1}=\varnothing$. As $V_i$ is a subspace containing $Z_i$, $i=1,2$, it follows that $V_i\cap S^{n-1}$ is a support for $\eta_i$, and thus $(V_1\cup V_2)\cap S^{n-1}$ is a support for $\eta^s=\eta_1^s+\eta_2^s$. Let $\bar\eta_i$ be the restriction of $\eta$ to $V_i\cap S^{n-1}$, $i=1,2$. It is enough to prove that $\bar\eta_i=\eta_i$. Obviously, $\bar\eta_i^s=\eta_i^s$ and thus by Lemma 2.1 $Z(\bar\eta_i)=Z(\eta_i)+X_i$, $i=1,2$. As $Z(\bar\eta_i)-Z(\eta_i)\subset V_i$, $X_i\in V_i$. However

$$Z(\eta)=Z(\eta_1)+Z(\eta_2)=Z(\bar\eta_1)+Z(\bar\eta_2)$$
$$=X_1+X_2+Z(\eta_1)+Z(\eta_2)=(X_1+X_2)+Z(\eta).$$

Therefore $X_1+X_2=0$. But $\dim(V_1+V_2)=\dim V_1+\dim V_2$ and therefore $X_1+X_2=0$ implies that $X_1=0=X_2$. Thus $Z(\bar\eta_i)=Z(\eta_i)$. As $Z(\eta_i)$ is definite, $\bar\eta_i=\eta_i$.

COROLLARY 3.5. *Every parallelepiped containing* 0 *is a definite zonoid.*

PROOF. Every one-dimensional zonoid is definite and a parallelepiped containing 0 is an independent sum of one-dimensional zonoids.

The class of zonoids obeying condition (C) of Theorem 1 will be denoted $\mathcal{F}_0$.

LEMMA 3.6. (D) $\Leftrightarrow$ (C).

PROOF. (C) $\Rightarrow$ (D). Let $Z$ be a zonoid in $\mathcal{F}_0$. We will prove that $Z$ is definite by induction on $\dim Z-\dim F(0)$ where $F(0)$ is the (unique) extreme set of $Z$ containing 0 in its relative interior. Without loss of generality (w.l.o.g.) $\dim Z=n$. If $\dim Z-\dim F(0)=0$, then 0 is in the relative interior of $Z$, and as $Z\in\mathcal{F}_0$, it means that $Z$ is a parallelepiped which is definite by Corollary 3.5. If $\dim Z-\dim F(0)\geqq 1$, there is $\xi\in R^n$ such that

$$0\in F(Z,\xi)=\{v\in Z\mid \langle v,\xi\rangle=\max\{\langle x,\xi\rangle\mid x\in Z\}.$$

As $F(Z, \xi)$ is an extreme set of $Z$, it follows that $F(0) \subset F(Z, \xi)$. Let $Z = Z(\mu)$, and let $\mu_1$ be the standard measure $\mu$ restricted to $\xi^{\perp} \equiv \langle x \in R^n \mid \langle x, \xi \rangle = 0\}$, and let $\mu_2 = \mu - \mu_1$. Then, $0 \in \operatorname{ext} Z(\mu_2)$ and $F(Z, \xi) = Z(\mu_1)$.

By Lemma 3.3, $Z(\mu_2)$ is definite, and by the induction hypothesis $Z(\mu_1)$ is definite. Thus for any standard measure $\sigma$ with $Z(\sigma) = Z$, if $\sigma_1$ denotes the standard measure $\sigma$ restricted to $\xi^{\perp}$, $Z(\sigma_1) = F(Z, \xi) = Z(\mu_1)$ and thus $\sigma_1 = \mu_1$, and $Z(\sigma - \sigma_1) = Z(\mu_2)$ and thus $\sigma - \sigma_1 = \mu_2$ which, together with $\sigma_1 = \mu_1$, implies that $\sigma = \mu$, i.e., that $Z$ is definite. This proves that (C) $\Rightarrow$ (D).

(D) $\Rightarrow$ (C). Assume that $Z$ is definite. W.l.o.g. $\dim Z = n$. First assume that 0 is in the relative interior of $Z$, i.e., that $F(0) = Z$. By Lemma 3.2, there is a zonoid decomposition $Z = Z_1 + Z_2$ where $0 \in \operatorname{ext} Z_1$, and 0 is the center of symmetry of $Z_2$. Obviously, if $Z$ is definite then $Z_2$ is definite and thus, by Lemma 3.1, $Z_2$ is a parallelepiped. As $0 \in \operatorname{ext} Z_2$, and 0 is in the relative interior of $Z_1$, it follows that $\dim F(0) = \dim Z_1$ and therefore $\dim Z_1 = n$. Let $x_1, \cdots, x_n$ be $n$ linearly independent vectors in $S^{n-1}$ such that $A = \{x_1, \cdots, x_n\} \cup \{-x_1, \cdots, -x_n\}$ is a support of $\mu_2$ where $Z_2 = Z(\mu_2)$. As 0 is the center of symmetry of $Z_2, \mu_2(\{x_i\}) = \mu_2(\{-x_i\}) > 0$. Let $Z_1 = Z(\mu_1)$. In order to prove that $Z_1 + Z_2$ is a parallelepiped it is enough to show that $A$ is a support of $\mu_1$. Otherwise, there is $B \in \mathscr{B}$ such that $B \cap A = \emptyset$, $B \cap -B = \emptyset$ and $\tilde{\mu}_1(B) \neq 0$. As $x_1, \cdots, x_n$ are linearly independent, there is a linear combination $\Sigma \alpha_i x_i = \tilde{\mu}_1(B)$. Let $0 < \beta \leqq 1$ be such that $|\beta\alpha_i| \leqq \mu_2(\{x_i\})$. Let $\mu_4$ be the standard measure supported on $A$ and given by

$$\mu_4(\{-x_i\}) - \beta\alpha_i = \mu_2(\{-x_i\}) = \mu_2(\{x_i\}) = \mu_4(\{x_i\}) + \beta\alpha_i.$$

Let $\mu_3$ be the standard measure given by

$$\mu_3(E) = \mu_1(E) - \beta \cdot \mu_1(E \cap B) + \beta\mu_1(-E \cap B).$$

Observe that $\sigma = \mu_3 + \mu_4$ is a standard measure, and that $\sigma \neq \mu$. It could be easily verified that for every $E \in \mathscr{B}$, $\sigma(E \cup -E) = \mu(E \cup -E)$ and that $\tilde{\sigma}(S^{n-1}) = \tilde{\mu}(S^{n-1})$ and therefore $Z(\sigma) = Z(\mu)$. Thus by contradiction $A$ is a support for $\mu_1$, which completes the proof in the case $Z = F(0)$. Otherwise, $0 \in \operatorname{ext} Z$, and thus there is $\xi \in R$ such that $0 \in F(Z, \xi)$. Let $Z = Z(\mu)$ and let $\mu_1$ be the standard measure $\mu$ restricted to $\xi^{\perp}$. Then $Z(\mu_1)$ is definite and thus by the induction hypothesis, the extreme set of $F(Z, \xi)$ containing 0 in its relative interior is a parallelepiped. This completes the proof of Lemma 3.6.

LEMMA 3.7. (D) $\Leftrightarrow$ (B).

PROOF. (D) $\Rightarrow$ (B). Let $Z$ be a definite zonoid and let $Z = Z(\eta)$ where $\eta$ is a standard measure. Let $(X, \Sigma, \mu)$ be the product vector measure space $(S^{n-1}, \mathscr{B}, \bar{\eta}) \times ([0,1], \mathscr{B}, \lambda)$ where $\lambda$ is Lebesgue measure on $([0,1], \mathscr{B})$. Let $Z = Z_1 + Z_2$ be a zonoid decomposition of $Z$. Let $Z_1 = Z(\eta_1)$, $Z_2 = Z(\eta_2)$ where $\eta_1$ and $\eta_2$ are standard measures. Then $Z(\eta) = Z = Z_1 + Z_2 = Z(\eta_1) + Z(\eta_2) = Z(\eta_1 + \eta_2)$ and as $Z$ is definite, $\eta = \eta_1 + \eta_2$. Let $f_1$ be the Radon–Nikodym derivative of $\eta_1$ with respect to $\eta$; then $0 \leqq f \leqq 1$. Let $S = \{(u,t) \mid u \in S^{n-1}, 0 \leqq t \leqq f(u)\}$, then $S$ is measurable and $\mathscr{R}(\mu, S) = Z(\eta_1)$, which completes the proof that (D) $\Rightarrow$ (B).

(B) $\Rightarrow$ (D). By contradiction. Assume that $Z$ is not a definite zonoid. Let $(X, \Sigma, \mu)$ be a non-atomic vector measure with $\mathscr{R}(\mu) = Z$. Let $\eta_\mu$ be the standard measure associated to $\mu$; and let $\eta$ be a standard measure with $\eta \neq \eta_\mu$ and $Z(\eta) = Z = Z(\eta_\mu)$. As $\eta \neq \eta_\mu$, there are $\xi \in R^n$, $\alpha > 0$ for which $(\eta - \eta_\mu)(\{u \in S^{n-1} \mid \langle u, \xi \rangle \geqq \alpha\}) > 0$. Let $A = \{v \in S^{n-1} \mid \langle v, \xi \rangle \geqq \alpha\}$, and let $\eta_1$ be the standard measure $\eta$ restricted to $A$. Then $Z = Z(\eta_1) + Z(\eta - \eta_1)$. Assume that there is $S \in \Sigma$ with $\mathscr{R}(\mu, S) = Z(\eta_1)$. Obviously the standard measure $\eta'$ associated to $\mu_{|S}$ obeys $\eta'(A) \leqq \eta_\mu(A)$. However, 0 is an extreme point of $Z(\eta_1)$ and thus $\eta_1 = \eta'$, which contradicts the inequalities $\eta'(A) \leqq \eta_\mu(A) < \eta_1(A)$.

Lemma 3.8. (D) $\Leftrightarrow$ (A).

Proof. (D) $\Rightarrow$ (A). Let $Z$ be a definite zonoid and let $\mu, \sigma$ be two non-atomic vector measures on the measurable spaces $(X_1, \Sigma_1)$, $(X_2, \Sigma_2)$, respectively. Let $(x_1, \cdots, x_m) \in \mathscr{R}_m(\mu)$ and let $(E_i)_{i=1}^m$ be a measurable partition of $X_1$ (i.e., $E_i \in \Sigma_1$, $i \neq j \Rightarrow E_i \cap E_j = \emptyset$, $\bigcup_{i=1}^n E_i = X_1$) with $\mu(E_i) = x_i$. Let $\eta$ be the standard measure with $Z(\eta) = \mathscr{R}(\mu)$ and let $\eta_i$ be the standard measure associated to the vector measure $\mu$ restricted to $E_i$. Then $\Sigma_{i=1}^m \eta_i = \eta$. Let $f_i$, $1 \leqq i \leqq m$, be the Radon–Nikodym derivative of $\eta_i$ with respect to $\eta$. Let $g$ be the Radon–Nikodym derivative of $\sigma$ with respect to $|\sigma|$, and let $\bar{f}_i = f_i \circ g$. Then $0 \leqq \bar{f}_i \leqq 1$, $\Sigma_{i=1}^m \bar{f}_i \leqq 1$, and $\int \bar{f}_i d\sigma = \int f_i(v) v d\eta = X_i$. By [3, theorem 4], there is a measurable partition $(S_i)_{i=1}^m$ of $(X_2, \Sigma_2)$ for which $\sigma(S_i) = x_i$. Therefore $(x_1, \cdots, x_m) \in \mathscr{R}_m(\sigma)$ which completes the implication (D) $\Rightarrow$ (A).

(A) $\Rightarrow$ (D). By contradiction. Assume that $Z$ is not definite. Let $\eta_1, \eta_2$ be two different standard measures with $Z(\eta_1) = Z = Z(\eta_2)$. Let $\xi \in Z$ be given by $\|\xi\| = \max\{\|x\| : x \in Z\}$. Then, $F(Z, \xi) = \{\xi\}$ and thus both $\eta_1$ and $\eta_2$ are supported on $S^{n-1} \setminus \xi^\perp$ and $\xi = \int_{H_\xi^+} u d\eta_i(u)$, $i = 1, 2$. Let $\bar{\eta}_i$ be the standard measure $\eta_i$ restricted to $H_\xi^+$. As $Z(\eta_1) = Z(\eta_2)$, $\eta_1^s = \eta_2^s$ (Lemma 2.1) and therefore if $\eta_1 \neq \eta_2$ then also $\bar{\eta}_1 \neq \bar{\eta}_2$. But $0 \in \operatorname{ext} Z(\bar{\eta}_1)$ and thus $Z(\bar{\eta}_1) \neq Z(\bar{\eta}_2)$. Without loss of generality there is $x \in Z(\bar{\eta}_1) \setminus Z(\bar{\eta}_2)$. Let $y$ be the center of symmetry of $Z$, i.e., $2y = \bar{\eta}_i(S^{n-1})$. Then the vector $(x, \xi - x, 2y - \xi)$ is in

$\mathcal{R}_3(\tilde{\eta}_1 \times \lambda)$ but not in $\mathcal{R}_3(\tilde{\eta}_2 \times \lambda)$, where $\lambda$ is the Lebesgue measure on $[0,1]$. As $\mathcal{R}(\tilde{\eta}_2 \times \lambda) = Z = \mathcal{R}(\tilde{\eta}_1 \times \lambda)$ this contradicts (A).

COROLLARY 3.9. *Let $\mu$ and $\sigma$ be the two non-atomic vector measures. Then the following conditions are equivalent.*

(3.10) $$\mathcal{R}_m(\mu) = \mathcal{R}_m(\sigma) \qquad \textit{for every } m \geqq 1,$$

(3.11) $$\mathcal{R}_k(\mu) = \mathcal{R}_k(\sigma) \qquad \textit{for some } k \geqq 3,$$

(3.12) $$\mathcal{R}_3(\mu) = \mathcal{R}_3(\sigma),$$

(3.13) $$\eta_\mu = \eta_\sigma.$$

PROOF. The implication (3.10) $\Rightarrow$ (3.11) is trivial. For (3.11) $\Rightarrow$ (3.12) observe that if $\mathcal{R}_{k+1}(\mu) = \mathcal{R}_{k+1}(\sigma)$ then $\mathcal{R}_k(\mu) = \mathcal{R}_k(\sigma)$. (3.12) $\Rightarrow$ (3.13) was actually proved in (A) $\Rightarrow$ (D) of Lemma 3.8 and (3.13) $\Rightarrow$ (3.10) was proved in (D) $\Rightarrow$ (A) of Lemma 3.8.

## 4. Decompositions of sums of countable many one-dimensional zonoids

Every convergent countable sum of closed intervals containing 0 is a zonoid; this class of zonoids is denoted by $\mathcal{F}_a$.

THEOREM 4.1. *The following conditions on a zonoid $Z$ are equivalent:*

(4.2) $$Z \in \mathcal{F}_a,$$

(4.3) $Z = \alpha Z + (1-\alpha)Z, 0 \leqq \alpha \leqq 1$, *is a zonoid decomposition of* $Z$ *w.r.t. any non-atomic vector measure with range* $Z$.

THEOREM 4.4. *The following conditions on a zonoid $Z$ are equivalent:*

(4.5) $$Z \in \mathcal{F}_a \cap \mathcal{F}_0,$$

(4.6) *any zonoid decomposition* $Z = Z_1 + Z_2$ *is a decomposition w.r.t. any non-atomic vector measure with range* $Z$.

LEMMA 4.7. *The following conditions on a zonoid $Z$ are equivalent:*

(4.2) $$Z \in \mathcal{F}_a,$$

(4.8) *$Z = Z(\eta)$ for some purely atomic standard measure $\eta$,*

(4.9) *every standard measure $\eta$ with $Z(\eta) = Z$ is purely atomic.*

Proof. (4.2) $\Rightarrow$ (4.8). Observe that if $0 \in [x, y] \equiv \{tx + (1-t)y : 0 \leqq t \leqq 1\}$ where $x, y \in R^n$ then $[x, y] = [0, x] + [0, y]$ and thus if $Z \in \mathcal{F}_a$, $Z$ is of the form $\sum_{i=1}^{\infty} \alpha_i [0, x_i]$ with $\alpha_i > 0$, $x_i \in S^{n-1}$ and $\sum_{i=1}^{\infty} \alpha_i < \infty$. Let $\eta_i$ be the standard measure supported on $\{x_i\}$ with $\eta_i(\{x_i\}) = \alpha_i$. Then $Z(\eta_i) = \alpha_i[0, x_i]$ and by putting $\eta = \sum_{i=1}^{\infty} \eta_i$, $\eta$ is purely atomic and $Z(\eta) = \sum_{i=1}^{\infty} Z(\eta_i) = Z$.

(4.8) $\Rightarrow$ (4.9). If $Z(\eta_1) = Z(\eta_2)$ then $\eta_1^s = \eta_2^s$, but $\eta_i$ is purely atomic iff $\eta_1^s$ is purely atomic. The implications (4.9) $\Rightarrow$ (4.8) $\Rightarrow$ (4.2) are obvious.

Lemma 4.10. *Let $(X, \Sigma, \mu)$ be a non-atomic vector measure space, with standard measure $\eta_\mu$ which is purely atomic. Then for any two standard measures $\eta_1, \eta_2$ with $\eta_1 + \eta_2 = \eta_\mu$, $Z(\eta_\mu) \equiv \mathcal{R}(\mu) = Z(\eta_1) + Z(\eta_2)$ is a decomposition w.r.t. $\mu$.*

Proof. Let $\{y_i\}_{i=1}^{\infty}$ be the support of $\eta \equiv \eta_\mu$, where $y_i \in S^{n-1}$ and $i \neq j \Rightarrow y_i \neq y_j$, and let $\alpha_i = \eta(\{y_i\})$. Then as $\eta_1 + \eta_2 = \eta$, $\eta_1(\{y_i\}) = t_i\alpha_i$ with $0 \leqq t_i \leqq 1$ and $\{y_i\}_{i=1}^{\infty}$ is a support of $\eta_1$. Therefore, $Z(\eta_1) = \sum_{i=1}^{\infty} t_i\alpha_i[0, y_i]$. Let $f = d\mu/d(|\mu|)$ and $X_i = f^{-1}(y_i)$. Then $X_i \in \Sigma$ and $X_i \cap X_j = \emptyset$ whenever $i \neq j$.

By Liapounoff's theorem there are measurable subsets $Y_i$ of $X_i$ with $\mu(Y_i) = t_i\mu(X_i) = t_i\alpha_i y_i$, and thus $\mathcal{R}(\mu, Y_i) = t_i\mathcal{R}(\mu, X_i) = t_i\alpha_i[0, x_i]$ and thus if $Y = \bigcup_{i=1}^{\infty} Y_i$, $\mathcal{R}(\mu, Y) = \sum_{i=1}^{\infty} \mathcal{R}(\mu, Y_i) = \sum_{i=1}^{\infty} t_i\alpha_i[0, y_i] = Z(\eta_1)$.

Proof of Theorem 4.1. (4.2) $\Rightarrow$ (4.3). Let $(X, \Sigma, \mu)$ be a vector measure space with range $\mathcal{R}(\mu)$ in $\mathcal{F}_a$. Let $\eta_\mu$ be the standard measure associated to $\mu$. By Lemma 4.7, $\eta_\mu$ is purely atomic. Let $0 \leqq \alpha \leqq 1$ and let $\eta_\alpha = \alpha\eta_\mu$. Then $Z(\eta_\alpha) = \alpha Z(\eta_\mu)$ and $\eta_\alpha + \eta_{1-\alpha} = \eta_\mu$. Thus by Lemma 4.10, $\mathcal{R}(\mu) = \alpha Z(\eta_\mu) + (1-\alpha)Z(\eta_\mu)$ is a decomposition w.r.t. $\mu$.

(4.3) $\Rightarrow$ (4.2). By contradiction. Let $Z$ be a zonoid, $Z \notin \mathcal{F}_a$. Let $M(Z)$ be the set of all standard measures $\eta$ with $Z(\eta) = Z$. Theorem 4.2 of [1] implies the existence of a constant $\rho(Z)$ for which $\eta(S^{n-1}) \leqq \rho(Z)$ whenever $\eta \in M(Z)$. Therefore $M(Z)$ is a bounded subset of $C(S^{n-1})^*$. It is easy to verify that $M(Z)$ is a closed subset of $C(S^{n-1})^*$ in the weak*-topology. Thus, Alaoglu's theorem ([5], 3.15) implies that $M(Z)$ is a compact subset of $C(S^{n-1})^*$. Obviously $M(Z)$ is also convex and therefore by the Krien–Milman theorem ([5], 3.21) $M(Z)$ has an extreme point. Let $\eta$ be an extreme point of $M(Z)$. By Lemma 4.7, $\eta$ is not purely atomic. Let $\eta = \eta_1 + \eta_2$ be a decomposition of $\eta$ into a purely atomic part $\eta_1$ and a non-atomic part $\eta_2$, and $S^{n-1} = S(\eta_1) \cup S(\eta_2)$ a disjoint decomposition of $S^{n-1}$ as the union of a support $S(\eta_1)$ of $\eta_1$ and $S(\eta_2)$ of $\eta_2$. Let $(X, \Sigma_1, \mu_1)$ be the (product) non-atomic vector measure space $(S(\eta_1), \mathscr{B}, \bar{\eta}_1) \times ([0,1], \mathscr{B}, \lambda)$. Observe that $(d\mu_1/d\,|\mu_1|)(x,t) = x$ and that the standard measure associated to $\mu_1$ is $\eta_1$. Let $(X_2, \Sigma_2, \mu_2)$ be the non-atomic vector measure space $(S(\eta_2), \mathscr{B}, \bar{\eta}_2)$. Observe that $(d\mu_2/d\,|\mu_2|)(x) = x$ and that the standard measure associated to $\mu_2$ is $\eta_2$. Let $(X, \Sigma, \mu)$ be the disjoint sum of the two non-atomic vector measure spaces $(X_i, \Sigma_i, \mu_i)$, $i = 1, 2$. That is, $X = X_1 \cup X_2$ and for $A \subset X$, $A \in \Sigma$ iff $A \cap X_i \in \Sigma_i$, $i = 1,2$ and then $\mu(A) = \mu_1(A \cap X_1) + \mu_2(A \cap X_2)$. Observe that $\eta$ is the standard measure associated to $(X, \Sigma, \mu)$. Let $Y \in \Sigma$ with $\mathscr{R}(\mu, Y) = \frac{1}{2}Z$. Denote $Y_i = Y \cap X_i$, $i = 1, 2$ and identify $Y_2$ as a subset of $S(\eta_2)$. Let $\tau$ be the vector measure $\mu$ restricted to $(Y, \Sigma_{|Y})$. Observe that $|\tau|$ is the restriction of $|\mu|$ to $Y$, and that $g \equiv d\tau/d\,|\tau|$ is the restriction of $f \equiv d\mu/d\,|\mu|$ to $Y$. Let $\sigma$ be the standard measure associated to $(Y, \Sigma_{|Y}, \tau)$. Then for measurable $A \subset Y_2$, $\sigma(A) = |\tau|\ (g^{-1}(A)) = |\tau|(A) = |\mu|(f^{-1}(A)) = \eta(A)$ and for measurable $A \subset S(\eta_2) \setminus Y_2$, $\sigma(A) = |\tau|(g^{-1}(A)) = |\tau|(A) = 0$. Therefore if $\eta_2 \neq 0$ then $2\sigma \neq \eta$.

For measurable $A \subset S^{n-1}$, $\sigma(A) = |\tau| \circ (g^{-1}(A)) = |\tau|(f^{-1}(A) \cap Y) \leqq |\mu| \circ f^{-1}(A) = \eta(A)$. Therefore $\eta - \sigma$ is a standard measure and $Z(\eta - \sigma) + Z(\sigma) = Z(\eta)$. On the other hand, as $Z(\sigma) = \mathscr{R}(\mu, Y) = \frac{1}{2}Z$, $Z(\sigma) + Z(\sigma) = Z(\eta)$ and thus $Z(\eta - \sigma) = Z(\sigma)$. Therefore, $2\eta - 2\sigma \in M(Z)$ and $2\sigma \in M(Z)$. As $\eta$ is an extreme point in $M(Z)$, $2\sigma = \eta$ which contradicts the inequality $2\sigma \neq \eta$.

Proof of Theorem 4.4. (4.5) $\Rightarrow$ (4.6). Let $Z \in \mathscr{F}_a \cap \mathscr{F}_0$, and let $(X, \Sigma, \mu)$ be a vector measure space with range $\mathscr{R}(\mu) = Z$. Let $\eta_\mu$ be the standard measure associated to $\mu$. Let $Z = Z_1 + Z_2$ be a zonoid decomposition and let $\eta_1, \eta_2$ be two standard measures with $Z_i = Z(\eta_i)$, $i = 1, 2$. Then $Z(\eta_1 + \eta_2) = Z(\eta_1) + Z(\eta_2) = Z = Z(\eta_\mu)$. As $Z \in \mathscr{F}_0$, $Z$ is definite by Lemma 3.6 and therefore $\eta_\mu = \eta_1 + \eta_2$ and thus by Lemma 4.10, $Z = Z_1 + Z_2$ is a decomposition of $Z$ w.r.t. $\mu$.

(4.6) $\Rightarrow$ (4.5). Follows by the implication (4.3) $\Rightarrow$ (4.2) of Theorem 4.1, and (B) $\Rightarrow$ (C) of Theorem I.

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